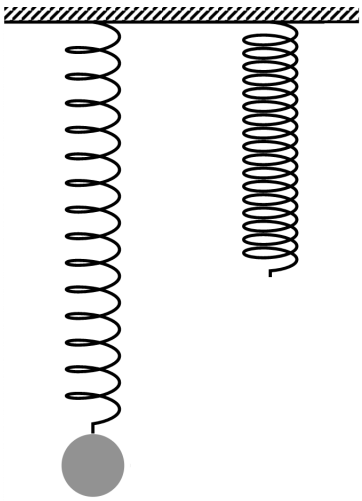
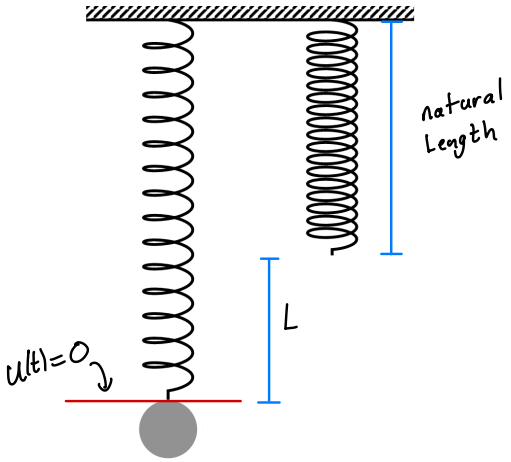
APMA 2130 - Pre-Class Worksheet: Spring-Mass Equations
Due: 8am on Mon, September 22, 2025

Suppose an object is attached to the end of a spring, and that $u(t)$ gives the position of the mass relative to an *equilibrium position* (discussed below). In this model, we assume that upward motion is *negative* and downward motion is *positive*.

1. Consider the picture for a spring-mass system.
   (a) One picture shows the spring at its *natural length*, that is, the length of the spring when nothing is attached to it.
   Label this length with "natural length"
   (b) The other picture shows the spring stretched by an object at the end of it. Here, the object at the end of the spring is stationary, and this is called the **equilibrium position**, $u(t) = 0$. Use L to denote the length the spring is stretched beyond its natural length.
   Label the appropriate length in the picture with "L", and the equilibrium position with "$u(t) = 0$".



Solution:



As the mass moves, we consider up to four different forces acting on it:

- Force from the spring, F_s which is proportional to the total length the spring is stretched.
- Force from damping (for example, air resistance), F_d which is proportional to velocity
- An external force, which we will denote by $F(t)$
- The force from gravity, which is constant.

2. (a) Give an expression for F_s, the force from the spring. This term should involve $u(t)$ or one of its derivatives.

Solution: We have $F_s = a[u(t) + L]$, where a is the constant of proportionality.

(b) When $u(t) = 0$, is the spring pulling the object up or pushing it down? What is the sign of F_s?

Solution: When $u(t) = 0$, the spring is pulling the object up. Thus, when $u(t) = 0$, we have that F_s is a negative number.

(c) What should the sign of the constant of proportionality be in F_s?

Solution: For $F_s = a \cdot u(t)$, we have that a is negative.

3. (a) Give an expression for F_d, the force from damping. This term should involve $u(t)$ or one of its derivatives.

Solution: F_d is proportional to velocity. Since velocity is the derivative of position, we have that F_d is proportional to $u'(t)$.
Thus, we have: $\boxed{F_d = b \cdot u'(t)}$

(b) If the object is moving up, what is the sign of the velocity and the sign of the force due to damping? What if the object is moving down?

Solution: Remember that downwards motion is considered positive. So for an upward moving object, the velocity would be negative, but the force due to damping would be positive, and for a downward moving object, the velocity would be positive, but the force due to damping would be negative.

(c) What should the sign of the constant of proportionality be?

Solution: We should have $F_d = b \cdot u'(t)$, where b is a negative number.

4. What is F_g, the force due to gravity? Remember that up is considered negative in this case.

Solution: $F_g = mg$

5. Newton's second law gives:

$$m \cdot a(t) = F_s(t) + F_d(t) + F_g(t) + F(t)$$

Use this, the relationship between position, acceleration, and velocity, where position is $u(t)$ to give a 2nd order differential equation for the position of the mass at time t.
Hint: What is acceleration in terms of position?

Solution: Since acceleration is the 2nd derivative of position, we can rewrite the equation as:

$$\boxed{m \cdot u''(t) = a[u(t) + L] + b \cdot u'(t) + F(t) + mg}$$