Names of Participating Group Members:	Group #:	Instructor/TA Signature:

APMA 2130 - Worksheet - Unit Step & Impulse Functions
Date: Thursday, October 23, 2025

Worksheet Objectives: After completing this worksheet, you should be able to do the following:

- Define and sketch the unit step and unit impulse functions.
- Write IVP forcing functions in terms of unit step and impulse functions.

Definitions:

- Consider the first and second order linear ODEs below with **forcing function** $G(t)$:

$$P(t)y' + R(t)y = G(t) \quad \text{and} \quad P(t)y'' + Q(t)y' + R(t)y = G(t)$$

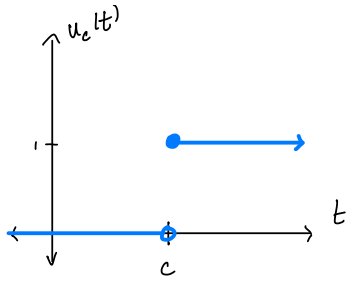
- The **unit step function** with a step at $t = c$ is denoted by $u(t-c)$ and is defined as:

$$u(t-c) = \begin{cases} 0 & t < c, \\ 1 & c \le t, \end{cases}$$

For the questions below, *well labeled* means that a scale for the vertical axis should be included, and the relevant t-values should be labeled.

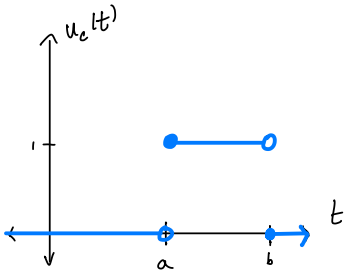
1. Sketch a well-labeled graph of the unit step function $u(t-c)$ for an arbitrary $c > 0$.

Solution:



2. Suppose $0 < a < b$. Sketch a well-labeled graph of $u(t-a) - u(t-b)$ below.

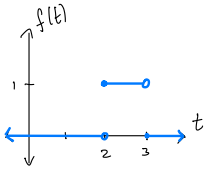
Solution:



3. Consider the piecewise defined function $f(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$

(a) Sketch a well-labeled graph of $f(t)$.

Solution:



(b) Write $f(t)$ using unit step functions.

Solution: $f(t) = u(t-2) - u(t-3)$

4. Consider the piecewise-defined function: $f(x) = \begin{cases} x^2 & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$

Write this using unit step functions.

Solution: $f(x) = x^2 g(x)$, where $g(x) = \begin{cases} 1 & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$.

From the work above, we have that $g(x) = u(x-4) - u(x-7)$.

Thus: $\boxed{f(x) = x^2[u(x-4) - u(x-7)]}$

5. Imagine that an object with a mass of 1kg is attached to the end of a spring with spring constant $k = 2$ and in a medium so that the damping coefficient $\gamma = 3$. Furthermore, suppose that the object is released at time $t = 0$ from a position of one meter below its equilibrium position.

Finally, suppose that initially, there is no external force on the object, but that between $t = 1$ and $t = 2$ a a downward force of 1 Newton is applied to it. After that, the system reverts again to having no external force.

(a) Set up an IVP for this situation. Use y for the solution variable instead of the usual u that we use for spring-mass equations, and give the forcing function as a **piecewise defined function**.

Solution: We have $mu'' + \gamma u' + ku = F(t)$, where $F(t)$ is the external force.

Here, we have $F(t) = \begin{cases} 1 & 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

Putting everything together, we get the IVP:

$$\boxed{y'' + 3y' + 2y = \begin{cases} 1 & 1 < t < 2 \\ 0 & \text{otherwise,} \end{cases} \qquad y(0) = 1,\ y'(0) = 0}$$

(b) Now, give the IVP where the forcing function is written using step functions instead of a piecewise defined function.

Solution:

$$\boxed{y'' + 3y' + 2y = u(t-1) - u(t-2) \qquad y(0) = 1,\ y'(0) = 0}$$

6. Suppose we have that $0 < \tau < 1$, and use the definition from the PCW: $d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$

(a) • Evaluate: $\int_{-3}^{3} d_\tau(t)\,dt$ • Determine the limit: $\lim_{\tau\to 0^+} \int_{-3}^{3} d_\tau(t)\,dt$

Solution: $\int_{-3}^{3} d_\tau(t)\,dt = 1$ and $\lim_{\tau\to 0^+} \int_{-3}^{3} d_\tau(t)\,dt = 1$

(b) Evaluate the limits. Feel free to use the provided QR code or desmos link to help. Note that a is used instead of τ in desmos.

$\lim_{\tau\to 0^+} d_\tau(0)$ $\lim_{\tau\to 0^+} d_\tau(t)$, for $t \neq 0$

Solution: $\lim_{\tau\to 0^+} d_\tau(t) = 0$, $\lim_{\tau\to 0^+} d_\tau(0) = \infty$

7. Consider function $f(t) = \int_0^t 4d_\tau(x-2)\,dx$, where $\tau = \frac{1}{10} = 0.1$.

(a) What is the value of $f(t)$ for $t = 1$? How does this generalize for $t \leq 1.9$?

Solution: If $t = 1$, then we have $f(t) = \int_0^1 4d_{0.1}(x-2)\,dx = \int_0^1 0\,dx = 0$. We also have that $f(t) = 0$ if $t \leq 1.9$. This is because $4d_{0.1}(x) = 0$ for all $x \leq 1.9$.

(b) What is the value of $f(t)$ for $t = 3.2$? How does this generalize for $2.1 \leq t$?

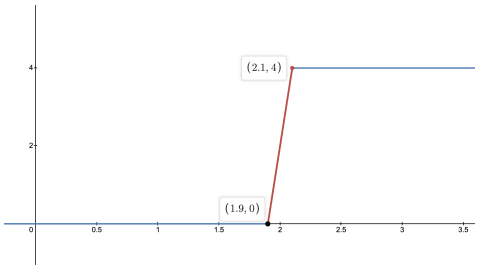
Solution: We have $f(3.2) = \int_0^{3.2} 4d_{0.1}(x-2)\,dx = \int_0^{1.9} 0\,dx + 4\int_{1.9}^{2.1} \frac{1}{2(0.1)}\,dx + \int_{2.1}^{3.2} 0\,dx = 4\int_{1.9}^{2.1} \frac{1}{2(0.1)}\,dx = 4(0.2)(5) = 4$. We also have that $f(t) = 4$ if $2.1 \leq t$.

(c) Take a guess at the graph of the function $f(t)$. Graph your guess below. Use the desmos link available through the QR code below to check your answers on this page and make any needed adjustments. Note that in desmos, a is used instead of τ.

Hint: You should already know the value of $f(t)$ for $t \leq 1.9$ and $t \geq 2.1$ from previous work.

Check your graph with an instructor or TA before moving on to the next page.

Solution:
Because we know $f(t) = 0$ for $t < 1.9$ and $f(t) = 4$ for $t > 2.1$, we know that the graph is at least the blue portion below. Hopefully, it makes sense that $f(t)$ should be continuous, which is where the red portion of the graph comes from.
We could also get this by evaluating the integral for an arbitrary t-value between $t = 1.9$ and $t = 2.1$.



Definition: The **unit impulse function** or **delta function** is denoted by $\delta(t)$, and can be thought of as $\delta(t) = \lim_{\tau \to 0^+} d_\tau(t)$, where $d_\tau(t)$ is the function defined in the pre-class worksheet. It satisfies:

- $\delta(t - t_0) = 0$ for all $t \neq t_0$
- $\displaystyle\int_a^b \delta(t - t_0)\, dt = 1$ when $a < t_0 < b$

We can take a to be $-\infty$ and b to be ∞.

8. Consider the function $f(t)$ defined by the integral $f(t) = \displaystyle\int_0^t [2\delta(x-1) - \delta(x-3)]\, dx$.

(a) What is the value of $f(t)$ for $t = 0.7$? How does this generalize to the value of $f(t)$ for $t < 1$?

Solution: For $t = 0.7$, $f(t)$ is the integral of a function that is always zero.
Thus, $f(0.7) = \int_0^{0.7} 0\, dt = 0$ and we have: $\boxed{f(t) = 0 \quad \text{for} \quad t < 1.}$

(b) What is the value of $f(t)$ for $t = 2.5$? How does this generalize to the value of $f(t)$ for $1 < t < 3$?

Solution: For $t = 2.5$, the integral defining $f(t)$ includes $x = 1$, but not $x = 3$.
Thus, $f(2.5) = 2\int_0^{2.5} \delta(x-1)\, dx = 2$ we have: $\boxed{f(t) = 2 \quad \text{for} \quad 1 < t < 3}$.